JOURNAL OF ALGEBRA 5, 250-265 (1967)

# Jordan Algebras and Lie Algebras of Type $D_4$*

HARRY P. ALLEN[1]

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts*

*Communicated by Nathan Jacobson*

Received January 20, 1966

In a recent paper, Jacobson [7] gave a new realization of the split Lie algebra $D_4$ based on Cayley algebras. Using this he was able to give a simple and explicit description of the group of automorphisms of the split $D_4$ and also to determine complete isomorphism conditions for algebras of type $D_{4I}$ and $D_{4II}$.[2] In this paper we shall apply the techniques introduced in [7] to the realization of the split Lie algebra $D_4$ of Chevalley and Schafer [2] in order to study $D_4$ forms. Using Galois cohomology of forms of algebras [6–8], we assign to each Lie algebra of type $D_4$ an element of order one or two of a second cohomology group, which we call the two-cohomology class of the algebra. The algebras with trivial two-cohomology class are determined and a simple condition is given for isomorphism. Several characterizations are given of these algebras and we use them to classify Lie algebras of type $D_4$ over finite and local fields (finite algebraic extensions of $p$-adic fields). Besides this we shall indicate some partial results for $D_4$'s over algebraic number fields.

In the first two sections we will describe the setting in which we work and introduce the machinery which will be employed. Much of this, as presented, is new and is based on the results of [7]. In several instances we have indicated how to translate the results and methods of [7] into a more usable form and no attempt has been made to present proofs. Thus we must assume a basic familiarity with [7] as well as a general knowledge both of Galois cohomology of forms of algebras ([7] suffices for this but [8] is more detailed) and of the properties of exceptional central simple Jordan algebras. Most of the Jordan algebra content can be found in [5].

* This research was supported by National Science Foundation Grant NSF-GP-4017.

[1] This paper is part of the authors dissertation which was submitted to the Graduate School of Yale University in 1965 in partial fulfillment of the requirements for the Ph.D. degree.

[2] These are the Lie algebras of type $D_4$ which have the form $\mathfrak{S}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra of degree 8 with involution.

In what follows, unless specific mention to the contrary is made, all fields are assumed to have characteristic 0 or $p \neq 2, 3$.

## 1. Jordan Algebras and the Lie Algebra $D_4$

Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \gamma)$ be a split exceptional central simple Jordan algebra over the field $P$. The Peirce decomposition of $\mathfrak{J}$ with respect to the diagonal idempotents $\{e_1, e_2, e_3\}$ is

$$\mathfrak{J} = \sum Pe_i \oplus \mathfrak{J}_{23} \oplus \mathfrak{J}_{31} \oplus \mathfrak{J}_{12}, \tag{1}$$

where $\mathfrak{J}_{ij} = \{a_{ij} = aE_{ij} + \gamma_j^{-1}\gamma_i \bar{a}E_{ij} \mid a \in \mathfrak{C}\}$. The elements of $\mathfrak{J}$ have the form

$$A = \begin{pmatrix} \alpha_1 & a_3 & \gamma_1^{-1}\gamma_3\bar{a}_2 \\ \gamma_2^{-1}\gamma_1\bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \gamma_3^{-1}\gamma_2\bar{a}_1 & \alpha_3 \end{pmatrix} \tag{2}$$

where $\alpha_i \in P$ and $a_i \in \mathfrak{C}$, the split Cayley algebra over $P$. The generic norm form on $\mathfrak{J}$ is given by

$$N(A) = \alpha_1\alpha_2\alpha_3 + \mathrm{Tr}((a_1a_2)\, a_3) - \sum \alpha_i \gamma_{i\theta^2}^{-1} \gamma_{i\theta}\, n(a_i), \tag{3}$$

where $n(a)$ is the norm form on $\mathfrak{C}$ and $\theta$ is the permutation (123).

$\mathfrak{J}$ is a natural representation space for three exceptional Lie algebras. The algebra $\mathfrak{L}(\mathfrak{J})$ of all norm skew transformations is the Lie algebra $E_6$ [*2, 4, 10*]. $\mathfrak{D}(\mathfrak{J})$, the derivation algebra of $\mathfrak{J}$, is the Lie algebra $F_4$ and is characterized as the subspace of $\mathfrak{L}(\mathfrak{J})$ annihilating $1 \in \mathfrak{J}$. The subalgebra of $\mathfrak{D}(\mathfrak{J})$ which maps $\{e_1, e_2, e_3\}$ onto 0 which we denote by $\mathfrak{D}(\mathfrak{J}/\sum Pe_i)$, is the Lie algebra $D_4$. In fact we have the sharper conclusion of

Theorem 1 (Chevalley and Schafer). *$\mathfrak{D}(\mathfrak{J}/\sum Pe_i) \cong \mathfrak{S}(\mathfrak{C})$, the Lie algebra of skew linear transformations in $\mathfrak{C}$ relative to the norm bilinear form.*

*Proof.* (Jacobson). Let $D \in \mathfrak{D}(\mathfrak{J}/\sum Pe_i) = \mathfrak{D}$. $e_iD = 0$, $i = 1, 2, 3$' implies $\mathfrak{J}_{ij}D \subseteq \mathfrak{J}_{ij}$ and we set $a_{ij}D = (aD_{ij})_{ij}$. By operating on the relation $a_{ij} \cdot b_{ij} = \gamma_j^{-1}\gamma_i n(a, b)(e_i + e_j)$ with $D$, we see that $D_{ij} \in \mathfrak{S}(\mathfrak{C})$ so $d_{ij} : D \rightarrow D_{ij}$ is a lie algebra homomorphism of $\mathfrak{D}$ into $\mathfrak{S}(\mathfrak{C})$. From the equation

$$2a_{ij} \cdot b_{jk} = (ab)_{ik}, \qquad i, j, k \text{ unequal},$$

we deduce the relation $(ab)\,\hat{D}_{23} = (aD_{31})\, b + a(bD_{12})$. Thus $D_{23}$, $D_{31}$, and $D_{12}$ are related by the principle of local triality ([*7*], Lemma 1; [*13*]), and we have $D_{31} = \hat{D}_{23}^{\sigma_1} = D_{23}^{\sigma_3\sigma_1}$ and $D_{12} = \hat{D}_{23}^{\sigma_2} = D_{23}^{\sigma_3\sigma_2}$. This shows that $D_{31}$ and $D_{12}$ are completely determined by $D_{23}$. Together with (1), this shows that $D$ is determined only by $D_{23}$ so $d_{23}$ is a monomorphism. If $L \in \mathfrak{S}(\mathfrak{C})$, set $D_{31} = \hat{L}^{\sigma_1}$, $D_{12} = \hat{L}^{\sigma_2}$ and define $D \in \mathrm{Hom}_{p}(\mathfrak{J}, \mathfrak{J})$ by $e_i D = 0$ and $D \mid \mathfrak{J}_{ij} = D_{ij}\,(D_{23} = L)$. Computations similar to the above but in reverse, show that $D \in \mathfrak{D}$ and that $d_{23} : D \to L$. Thus $d_{23}$, and similarly every $d_{ij}\, i \neq j$, is an isomorphism.

The preceeding proof enables us to display the action of any $D$ in $\mathfrak{D}$ explicitly. If $A$ is given by (2) and $D_{23} = L$, then

$$AD = \begin{pmatrix} 0 & a_3 L^{\sigma_1\sigma_3} & \gamma_1^{-1}\gamma_3 \bar{a}_2 L^{\sigma_2} \\ \gamma_2^{-1}\gamma_1 \bar{a}_3 L^{\sigma_1} & 0 & a_1 L \\ a_2 L^{\sigma_2\sigma_3} & \gamma_3^{-1}\gamma_2 \bar{a}_1 L^{\sigma_3} & 0 \end{pmatrix} \tag{4}$$

since $\sigma_2\sigma_3 = \sigma_3\sigma_1$ and $\sigma_i^2 = 1$. Also the Pierce spaces $\mathfrak{J}_{23}$, $\mathfrak{J}_{31}$, and $\mathfrak{J}_{12}$ are inequivalent $\mathfrak{D}$-modules, for (4) shows that they are equivalent, respectively, to $\mathfrak{C}$ as a $\mathfrak{S}(\mathfrak{C})$-module relative to the representations $L \to L$, $L \to L^{\sigma_2\sigma_3}$, and to $L \to L^{\sigma_1\sigma_3}$ (cf. [7] Lemma 1).

As a notational convenience we will write $A = \sum \alpha_i e_i + (a_i)$ instead of (2) and if $D \in \mathfrak{D}$ we will write $D = (D_i)$, $D_1 = D_{23}$, $D_2 = D_{31}$ and $D_3 = D_{12}$. With this notation, (4) becomes $AD = (a_i D_i)$.

Let $(p, T)$ be a projectivity of $\mathfrak{C}^{(3)}$ ([7], Definition 3). Define a transformation $[p, T]$ in $\mathfrak{J}$ by

$$A[p, T] = \sum \alpha_i x_i e_{ip-1} + (a_i)(p, T), \tag{5}$$

where

$$(a_i)(p, T) = \begin{cases} (a_{ip} T_i) & \text{if } p \text{ is even,} \\ (\bar{a}_{ip} T_i) & \text{if } p \text{ is odd,} \end{cases}$$

and

$$x_i = \lambda \mu_{ip^{-1}}^{-1} (\gamma_{ip^{-1}\theta^2} \gamma_{ip^{-1}\theta}^{-1})(\gamma_{i\theta^2}^{-1} \gamma_{i\theta}).$$

Here $\theta$ is the permutation (123), $\lambda$ is given by $(xT_1, yT_2, zT_3) = \lambda(x, y, z)$, $(x, y, z) = \frac{1}{2}\,\mathrm{tr}((xy)\,z)$ (Eq. (12) of [7]), and $\mu_i$ by $n(aT_i) = \mu_i n(a)$.

Lemma 1. *If* $[p, T]$ *is defined as above, then* $N(A[p, T]) = \lambda N(A)$ *for all* $A \in \mathfrak{J}$ *(*$\lambda$ *as above).*

*Proof.* This is a routine computation. One only needs the relation $x_1 x_2 x_3 = \lambda$ and this follows immediately from $\mu_1\mu_2\mu_3 = \lambda^2$. The latter is deduced from (13) of [7] by taking the norm of both sides.

The mappings $[p, T]$ form a group of norm equivalences in $\mathfrak{J}$ since we easily recover the composition formula ([7], p. 8)

$$[p, T][q, U] = [qp, T^q U].$$

By a norm equivalence of $\mathfrak{J}$ we mean a linear automorphism $C$ of $\mathfrak{J}$ which satisfies $N(AC) = \lambda N(A)$ for some $\lambda \in P^*$ and all $A$ in $\mathfrak{J}$. The mappings $[p, T]$ are clearly in the group $GL(\mathfrak{J}/\sum Pe_i)$ of all norm equivalences of $\mathfrak{J}$ which leave $\sum Pe_i$ stable. This group, as we shall now see, determines all $P$-linear automorphisms of $\mathfrak{D}$.

THEOREM 2. *Every $P$-linear automorphism of $\mathfrak{D}$ has the form $D \to C^{-1}DC$ for some $C$ in $GL(\mathfrak{J}/\sum Pe_i)$. Conversely, such a mapping leaves $\mathfrak{D}$ stable and induces a $P$-linear automorphism.*

*Proof.* If $C \in GL(\mathfrak{J}/\sum Pe_i)$ then $C^{-1}\mathfrak{D}C \subseteq \mathfrak{L}(\mathfrak{J})$ ([4] and [10]). But $\sum Pe_i$ is stable under $C^{-1}$ so $\sum Pe_i C^{-1}\mathfrak{D}C = 0$ and thus $C^{-1}\mathfrak{D}C \subseteq \mathfrak{D}$. Hence $D \to C^{-1}DC$ is a $P$-linear automorphism of $\mathfrak{D}$. The remainder of the proof is identical to that of ([7], Theorem 2), and is omitted.

If $\eta \in \mathrm{Aut}_P\, \mathfrak{D}$ has the form $D \to C^{-1}DC$ with $C \in GL(\mathfrak{J}/\sum Pe_i)$, then we shall write $\eta \leftrightarrow C$. (Note that this is a one–many correspondence). Theorem 2 shows that we have a natural homomorphism of $GL(\mathfrak{J}/\sum Pe_i)$ onto $\mathrm{Aut}_P\, \mathfrak{D}$ and we let $K$ denote the kernel. This gives rise to the exact sequence

$$1 \to K \to GL\left(\mathfrak{J}\Big/\sum Pe_i\right) \to \mathrm{Aut}_P\, \mathfrak{D} \to 1. \tag{6}$$

[*Note.* If $P$ is algebraically closed, or at least is closed under the taking of square roots, then this sequence can be considerably simplified. If $\mathrm{Aut}_P(\mathfrak{J}/\sum Pe_i)$ denotes the group of all automorphisms of $\mathfrak{J}$ which leave $\sum Pe_i$ stable, and $\mathfrak{K}$ denotes the subgroup of $K$ all of whose coordinates are $\pm 1$ with an even parity of $-1$'s (see Proposition 1 below), then $\mathfrak{K}$ is the Klein four-group and

$$1 \to \mathfrak{K} \to \mathrm{Aut}_P(\mathfrak{J}/\sum Pe_i) \to \mathrm{Aut}_P\, \mathfrak{D} \to 1$$

is exact.]

As in [7], Theorem 2, we have

PROPOSITION 1. $K \cong P^* \times P^* \times P^*$.

*Proof.* The correspondence, $(\delta_i) \to [1, (\delta_i)]$ is a monomorphism of $P^* \times P^* \times P^*$ in $K$. We will show that it is epimorphic.

Let $\mathfrak{D}^*$ be the $P$-subalgebra of $\mathrm{Hom}_P(\mathfrak{J}, \mathfrak{J})$ generated by $\mathfrak{D}$. $\mathfrak{J}$ is a faithful completely reducible $\mathfrak{D}^*$-module so $\mathfrak{D}^*$ is semisimple. Equation (1) is the decomposition of $\mathfrak{J}$ into homogeneous $\mathfrak{D}^*$-components and by a proof of the Artin–Wedderburn theorem, $\mathfrak{D}^* = \mathfrak{D}_1{}^* \oplus \mathfrak{D}_2{}^* \oplus \mathfrak{D}_3{}^*$, $\mathfrak{D}_i{}^*$ simple ideals isomorphic to $\mathrm{Hom}_P(\mathfrak{C}, \mathfrak{C})$.

If $\delta \in K$, then $A \to A\delta$ is a $\mathfrak{D}$-automorphism of $\mathfrak{J}$, hence also a $\mathfrak{D}^*$-automorphism. This implies that each of $\sum Pe_i$, $\mathfrak{J}_{23}$, $\mathfrak{J}_{31}$, and $\mathfrak{J}_{12}$ is stable under $\delta$. Since $\delta$ commutes with every element of $\mathfrak{D}$, $\delta \mid \mathfrak{J}_{ij}$, $i \neq j$, is a nonzero scalar—say, $\delta_1$ on $\mathfrak{J}_{23}$, $\delta_2$ on $\mathfrak{J}_{31}$, and $\delta_3$ on $\mathfrak{J}_{12}$. Write $e_i\delta = \sum \alpha_{ij}e_j$ and suppose that $N(A\delta) = \mu N(A)$ for all $A$ in $\mathfrak{J}$.

$N(e_i) = 0$ implies $N(e_i\delta) = 0$, so $e_i\delta \in Pe_k + Pe_l$ for some $k, l$. Suppose that $e_i\delta = \alpha e_k + \beta e_l$ with $\alpha\beta \neq 0$ and $k \neq l$. Then for any $j$, $e_j\delta \in Pe_k + Pe_l$, because otherwise, for some $\epsilon \in P$, we would have $0 = \mu N(e_i + \epsilon e_j) = N((e_i + \epsilon e_j)\,\delta) \neq 0$, an absurdity. But this is impossible since $(\sum Pe_j)\,\delta$ is three dimensional and $Pe_k + Pe_l$ is two dimensional. Thus we must conclude that $e_i\delta \in Pe_k$ for some $k$.

By checking the action of $\delta$ on $1_{23} + 1_{31} + 1_{12}$ and elements of the form $e_i + 1_{i\theta i\theta^2}$ $[\theta = (123)]$ one sees that $\mu = \delta_1\delta_2\delta_3$ and that $e_i\delta = (\delta_1\delta_2\delta_3)\,\delta_i^{-2}e_i$. This shows that $\delta = [1, (\delta_i)]$ and concludes the proof.

We will consider the isomorphism described in proposition 1 as an identification and will write $(\delta_i)$ instead of $[1, (\delta_i)]$.

Corollary. *Every elements of $GL(\mathfrak{J}/\sum Pe_i)$ has the form $[p, T]$ for some uniquely determined projectivity $(p, T)$.*

The results just indicated may be extended to determine $\mathrm{Aut}_\Phi\, \mathfrak{D}$ where $\Phi$ is a subfield of finite codimension and Galois in $P$. If we let $\Gamma L_\Phi(\mathfrak{J}/\sum Pe_i)$ be the group of all $s$-linear norm equivalences ($s \in G = g(P/\Phi)$, $N(AC) = \lambda N(A)^s$) of $\mathfrak{J}$ leaving $\sum Pe_i$ stable, then every $\Phi$-linear automorphism of $\mathfrak{D}$ has the form $D \to C^{-1}DC$ for some $C \in \Gamma L_\Phi(\mathfrak{J}/\sum Pe_i)$. Conversely, any mapping of this form is a $\Phi$-linear automorphism of $\mathfrak{D}$. The sequence

$$1 \to K \to \Gamma L_\Phi\left(\mathfrak{J}\Big/\sum Pe_i\right) \to \mathrm{Aut}_\Phi\, \mathfrak{D} \to 1 \tag{7}$$

is exact and every element of $\Gamma L_\Phi(\mathfrak{J}/\sum Pe_i)$ has the form $[p, T]$ where $(p, T)$ is a $\Phi$-linear colliniation of $\mathfrak{C}^{(3)}$. Here if $(p, T)$ is an $s$-linear collineation of $\mathfrak{C}^{(3)}$, then $[p, T]$ is defined by $A[p, T] = \sum \alpha_i{}^s x_i e_{ip^{-1}} + (a_i)(p, T)$ where $x_i = \lambda\mu_{ip^{-1}}^{-1}(\gamma_{ip^{-1}\theta^2}\gamma_{ip^{-1}\theta}^{-1})(\gamma_{i\theta^2}^{-1}\gamma_{i\theta})^s$.

We note that neither of the sequences (6) or (7) is central, for if $\delta = (\delta_i) \in K$ and $[p, T] \in \Gamma L_\Phi(\mathfrak{J}/\sum Pe_i)$ is $s$-linear, then $(\delta_i)[p, T] = [p, T](\delta_{ip}^s)$.

## 2. Generalities

Throughout this section $\mathfrak{L}$ will denote a Lie algebra of type $D_4$ over a field $\Phi$ and $P/\Phi$ will be a sufficiently large finite-dimensional Galois splitting extension. We take $\mathfrak{L}_P = \mathfrak{D} = \mathfrak{D}(\mathfrak{J}/\sum Pe_i)$, the split Lie algebra $D_4$ of the preceeding section. It is easy to see that $\mathfrak{L}$ is the fixed point set in $\mathfrak{D}$ of a group of semi-automorphisms of $\mathfrak{D}$. More precisely, we have a homomorphism $s \to \eta(s)$ of $G = g(P/\Phi)$ into $\mathrm{Aut}_\Phi\, \mathfrak{D}$ such that $\eta(s)$ is $s$-linear and

$\mathfrak{L} = \{D \in \mathfrak{D} \mid D^{\eta(s)} = D \text{ for all } s \in G\}$. Such a homomorphism is called a precocycle of $G$ in $\mathrm{Aut}_\Phi \mathfrak{D}$. Conversely, it is well known [6–8] that the fixed-point set of a precocycle of $G$ in $\mathrm{Aut}_\Phi \mathfrak{D}$ is a $\Phi$-form of $\mathfrak{D}$.[3]

If $\eta(s) \leftrightarrow C(s) = [p(s), T(s)] \in \Gamma L_\Phi(\tilde{\mathfrak{J}}/\sum Pe_i)\,(\eta(s) : D \to C^{-1}(s)\,DC(s))$ then the composition formula $[p, T][q, U] = [qp, T^qU]$ for the latter mappings shows that $p : s \to p(s)$ is an antihomorphism of $G$ into $S_3$, the symmetric group on three letters. We call $p$ the antihomomorphism of $G$ associated with $\mathfrak{L}$ and say that $\mathfrak{L}$ is of type $D_{4\mathrm{I}}$, $D_{4\mathrm{II}}$, $D_{4\mathrm{III}}$, or $D_{4\mathrm{VI}}$, depending on whether $|p(G)| = 1, 2, 3$, or $6$. The $D_4$ type is independent of the isomorphism class of $\Phi$-forms to which $\mathfrak{L}$ belongs, as well as the choice of a Galois splitting extension ([7], Section 3, p. 12). $\mathfrak{L}$ is a special Lie algebra, i.e., $\mathfrak{L} = \mathfrak{S}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra of degree 8 with involution, if and only if $\mathfrak{L}$ is of type $D_{4\mathrm{I}}$ or $D_{4\mathrm{II}}$ ([7], Section 4). In the present context this can be seen as follows.

Let $\mathfrak{D}^*$ be the $P$-enveloping algebra of $\mathfrak{D}$ which was introduced in the proof of Proposition 1. $\mathfrak{D}^*$ has an involution $J$ whose restriction to each simple component of $\mathfrak{D}^*$ is the adjoint mapping in $\mathrm{Hom}_P(\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}})$ determined by the norm bilinear form on $\tilde{\mathfrak{C}}$. We let $\mathfrak{L}^*$ be the $\Phi$-subalgebra of $\mathrm{Hom}_P(\tilde{\mathfrak{J}}, \tilde{\mathfrak{J}})$ generated by $\mathfrak{L}$. Then $\mathfrak{L}^*$ is a $\Phi$-form of $(\mathfrak{D}^*, J)$ [it is the fixed point set of $\eta(s) : D^* \to C(s)^{-1}\,D^*C(s), D^* \in \mathfrak{D}^*$], and if we use $J$ for $J \mid \mathfrak{L}^*$, then $\mathfrak{L} \subseteq \mathfrak{S}(\mathfrak{L}^*, J)$. $(\mathfrak{L}^*, J)$ is unique up to isomorphism and is independent of the choice of Galois splitting extension. With these notions one may then prove

THEOREM 3 (Jacobson). (i) *If $\mathfrak{L}$ is of type $D_{4\mathrm{I}}$ then*

$$(\mathfrak{L}^*, J) = (\mathfrak{A}_1, J_1) \oplus (\mathfrak{A}_2, J_2) \oplus (\mathfrak{A}_3, J_3)$$

*where $(\mathfrak{A}_i, J_i)$ is a central simple associative algebra of degree 8 with involution. $\mathfrak{L} \cong \mathfrak{S}(\mathfrak{A}_i, J_i)$ and $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{A}_3 \sim 1$ in the Brauer group of $P/\Phi$.*

(ii) *If $\mathfrak{L}$ is of type $D_{4\mathrm{II}}$ then $(\mathfrak{L}^*, J) = (\mathfrak{A}, J) \oplus (\mathfrak{B}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra of degree 8 with involution and $(\mathfrak{B}, J)$ is a simple algebra with involution whose center is a quadratic field extension of $\Phi$. Moreover, $\mathfrak{L} \cong \mathfrak{S}(\mathfrak{A}, J)$.*

THEOREM 4 (Jacobson). (i) *A special Lie algebra of type $D_4$ is of type $D_{4\mathrm{I}}$ or $D_{4\mathrm{II}}$.*

(ii) *If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are special Lie algebras of type $D_4$ over $\Phi$, then $\mathfrak{L}_1 \cong \mathfrak{L}_2$ if and only if $(\mathfrak{L}_1{}^*, J) = (\mathfrak{L}_2{}^*, J)$.*

---

[3] This is a special case of the more general situation where $\mathfrak{D}$ is an arbitrary nonassociative algebra over $P$. Then precocycles of $G$ in $\mathrm{Aut}_\Phi \mathfrak{D}$ are defined as above and the analogous statements about $\Phi$-forms and precocycles are true.

If $\mathfrak{L}$ is exceptional, i.e., of type $D_{4\text{III}}$ or $D_{4\text{VI}}$, then one can extend Theorem 3 by showing that $\mathfrak{L}^*$ is simple. The center of $\mathfrak{L}^*$ is a cubic field extension of $\Phi$ and is Galois over $\Phi$ if and only if $\mathfrak{L}$ is of type $D_{4\text{III}}$. The second part of Theorem 4 is not true if $\mathfrak{L}$ is exceptional.

Again let $p$ be the antihomomorphism of $G$ into $S_3$ determined by $\mathfrak{L}$. If $H$ is the kernel of $p$, then $H$ is a normal subgroup of $G$ and hence its fixed field, $F$, is a Galois extension of $\Phi$. $g(F/\Phi) \cong G/H$ and may be identified with a subgroup of $S_3$. $\mathfrak{L}_F$ is clearly of type $D_{4\text{I}}$ and $F$ may be characterized (within a given algebraic closure of $\Phi$) as the smallest overfield of $\Phi$ over which $\mathfrak{L}$ is of type $D_{4\text{I}}$. We call $F$ the canonical $D_{4\text{I}}$-field extension of $\mathfrak{L}$ and $\mathfrak{L}_F$ the canonical $D_{4\text{I}}$ extension of $\mathfrak{L}$. Similarly if $\mathfrak{L}$ is of type $D_{4\text{VI}}$ and $F$ is its canonical $D_{4\text{I}}$-field extension, then there is a unique quadratic subextension of $F/\Phi$ over which $\mathfrak{L}$ is of type $D_{4\text{III}}$, and also three (conjugate) cubic subextensions of $F/\Phi$ over which $\mathfrak{L}$ is of type $D_{4\text{II}}$. These fields will be called, respectively, the canonical $D_{4\text{III}}$-field extension of $\mathfrak{L}$ and a canonical $D_{4\text{II}}$-field extension of $\mathfrak{L}$. The algebras obtained by extending the base field of $\mathfrak{L}$ will be called, respectively, the canonical $D_{4\text{III}}$ extension of $\mathfrak{L}$ and a canonical $D_{4\text{II}}$ extension of $\mathfrak{L}$.

Using the notation of Theorem 3 and its extension, one sees that if $\mathfrak{L}$ is of type $D_{4\text{II}}(D_{4\text{III}})$ then the center of $\mathfrak{B}(\mathfrak{L}^*)$ is the canonical $D_{4\text{I}}$-field extension of $\mathfrak{L}$, while if $\mathfrak{L}$ is of type $D_{4\text{VI}}$ then the center of $\mathfrak{L}^*$ is a canonical $D_{4\text{II}}$-field extension of $\mathfrak{L}$.

We note one additional property of $\mathfrak{L}^*$ which will be used in the sequel. If $\mathfrak{L}$ is exceptional, i.e., $\mathfrak{L}^*$ is simple, and $F$ is the canonical $D_{4\text{I}}$-field extension of $\mathfrak{L}$, then $\mathfrak{L}_F{}^* = (\mathfrak{L}^*)_F = (\mathfrak{A}_1, J_1) \oplus (\mathfrak{A}_2, J_2) \oplus (\mathfrak{A}_3, J_3)$ as in Theorem 3. If $s \in g(F/\Phi)$, $s^3 = 1$, $s \neq 1$ then the $s$-automorphism of $\mathfrak{L}_F{}^*$ which leaves $\mathfrak{L}^*$ fixed, permutes the factors of $\mathfrak{L}_F{}^*$ cyclically. Thus the $(\mathfrak{A}_i, J_i)$ are all ring-isomorphic. In particular, if one of them is a total matrix algebra then they all are.

As remarked earlier, the sequence (6) is not central and hence is unsuited for an application of Galois cohomology. We will now show how this obstruction may be circumvented. Let $GL(\mathfrak{J}/\{Pe_i\}_i)$ be the subgroup of $GL(\mathfrak{J}/\sum Pe_i)$ leaving each $Pe_i$ stable, i.e., all $[p, T]$ with $p = 1$. If we let $\mathrm{Aut}_P{}' \mathfrak{D}$ be the corresponding subgroup of $\mathrm{Aut}_P \mathfrak{D}$, then we obtain

$$1 \to K \to GL(\mathfrak{J}/\{Pe_i\}_i) \to \mathrm{Aut}_P{}' \mathfrak{D} \to 1, \tag{8}$$

a central exact sequence. Let $p$ be any antihomomorphism of $G$ into $S_3$, $\mathfrak{C}$ the split Cayley algebra over $\Phi$ and $s \to S$ the precocycle of $G$ in $\mathrm{Aut}_\Phi \tilde{\mathfrak{C}}$ corresponding to $\mathfrak{C}$. Then $(S, S, S) = S$ is a related triple of similarities and we define

$$C_p(s) = [p(s), S] \tag{9}$$

and $\eta_p(s)$ by,

$$\eta_p(s) \leftrightarrow C_p(s). \tag{10}$$

$C_p(st) = C_p(s)\, C_p(t)$ for all $s, t$ in $G$ so $\eta_p(st) = \eta_p(s)\, \eta_p(t)$ and $s \to \eta_p(s)$ is a precocycle of $G$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$.[4] (See [7], p. 12.) If $\beta \in \mathrm{Aut}_P\, \mathfrak{D}$ and $B \in GL(\mathfrak{J}/\sum Pe_i)$ define $\beta^s = \eta_p(s)^{-1}\, \beta\eta_p(s)$ and $B^s = C_p(s)^{-1}\, BC_p(s)$. These definitions equip each of the groups in (6) with a $G$-group structure and (6) is an exact sequence of $G$-groups and $G$-homomorphisms. The groups in (8) are all $G$-subgroups, so (8) too is an exact sequence of $G$-groups and $G$-homomorphisms. Since (8) is central we obtain the exact sequence of cohomology sets

$$H_p^1(G, K) \to H_p^1(G, GL(\mathfrak{J}/\{Pe_i\}_i)) \to H_p^1(G, \mathrm{Aut}_P{}'\, \mathfrak{D}) \xrightarrow{\Delta} H_p^2(G, K). \tag{11}$$

Here the subscript $p$ is used to indicate the generation of the $G$-action, $H_p^2(G, K)$ is the ordinary group-theoretic second cohomology group since $K$ is Abelian, and $\Delta$ is the coboundary mapping (cf. [11], Chap. VII, Annexe).

If $s \to \eta(s)$ is the precocycle of $G$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ corresponding to the $\Phi$-form $\mathfrak{L}$, and $p$ is the antihomomorphism of $G$ into $S_3$ determined by $\mathfrak{L}$, then by using the $G$-action afforded by $p$ as above, we obtain the 1-cocycle $s \to \gamma(s) = \eta_p(s)^{-1}\, \eta(s)$ of $G$ in $\mathrm{Aut}_P\, \mathfrak{D}$ corresponding to $\mathfrak{L}$. By our choice of $G$-action this 1-cocycle has values in $\mathrm{Aut}_P{}'\, \mathfrak{D}$ and we use $[\gamma]_p$ to denote its 1-cohomology class in $H_p^1(G, \mathrm{Aut}_P{}'\, \mathfrak{D})$. We define

$$\Gamma_P(\mathfrak{L}) = [\gamma]_p\, \Delta \in H_p^2(G, K)$$

and call it the two-cohomology class of $\mathfrak{L}$.

By a theorem of Springer ([12], Appendix) $H_p^1(G, K) = 1$. We shall see later that the elements of $H_p^1(G, GL(\mathfrak{J}/\{Pe_i\}_i)$ correspond to certain norm equivalence classes of $\Phi$-forms of isotopes of $\mathfrak{J}$ (Theorem 6). The elements of $H_p^2(G, K)$ correspond to certain equivalence classes of separable associative algebras over $\Phi$ split by $P$ and play a role which generalizes the relationship between factor sets and central simple associative algebras over $\Phi$ split by $P$. In this abstract correspondence $\Gamma_P(\mathfrak{L})$ corresponds to $\mathfrak{L}^*$ (see Section 5). The condition $\Gamma_P(\mathfrak{L}) = 1$ is independent of the isomorphism class of $\Phi$-forms to which $\mathfrak{L}$ belongs and since the inflation homomorphism is 1–1, it is also independent of the choice of Galois splitting field.

Finally we shall show that $\Gamma_P(\mathfrak{L})^2 = 1$. To see this we must first compute $\Gamma_P(\mathfrak{L})$ explicitly. If $s \to \eta(s) \leftrightarrow C(s) \in \Gamma L(\mathfrak{J}/\sum Pe_i)$ is the precocycle of $G$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ corresponding to $\mathfrak{L}$, then $\eta(st) = \eta(s)\, \eta(t)$ implies $C^{-1}(st)\, C(s)\, C(t) \in K$ and we set $C(s)\, C(t) = C(st)\, \delta_{s,t}$, $\delta_{s,t} \in K$. $(s, t) \to \delta_{s,t}$ is a two cocycle of $G$ in $K$ and $\Gamma_P(\mathfrak{L})$ is its two-cohomology class. [Here the action of $G$ on

[4] The $\Phi$-form corresponding to this precocycle is called a Steinberg $D_4$.

$K$ is just $(\delta_i)^s = C(s)^{-1}(\delta_i)\, C(s) = (\delta^s_{ip(s)})]$. We define the ratio map $\zeta : \Gamma L(\mathfrak{J}/\sum Pe_i) \to K$ by $\zeta([p, T]) = (\mu_i)$ where $\mu_i$ is the ratio of semisimilarity $T_i$. If we set $\rho_s = \zeta(C(s))$ then $C(s)\, C(t) = C(st)\, \delta_{s,t}$ implies $\rho_{st}^{-1}\rho^t{}_s\rho_t = \delta^2_{s,t}$, which is the desired relation.

## 3. Jordan $D_4$'s

Let $\mathfrak{J}$ be an exceptional central simple Jordan algebra over a field $\Phi$ and let $\mathfrak{k}$ be a cubic semisimple associative subalgebra of $\mathfrak{J}$. $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$, the subalgebra of the derivation algebra of $\mathfrak{J}$ which maps $\mathfrak{k}$ onto 0, is a Lie algebra of type $D_4$ since $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})_\Omega = \mathfrak{D}(\mathfrak{J}_\Omega/\mathfrak{k}_\Omega)$ for $\Omega$ the algebraic closure of $\Phi$.

Definition. A Lie algebra $\mathfrak{L}$ of type $D_4$ is called a Jordan $D_4$ if $\mathfrak{L} \cong \mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ for some $\mathfrak{J}$ and $\mathfrak{k}$ as above.

This section is devoted to the proofs of the equivalence of (1), (2), and (3) of Theorem I, and Theorem II.

Theorem I. *Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over a field $\Phi$ and $P/\Phi$ a finite-dimensional Galois splitting extension.* Then the following statements are equivalent.

(1) *$\mathfrak{L}$ is a Jordan $D_4$.*

(2) $\Gamma_P(\mathfrak{L}) = 1$.

(3) *The canonical $D_{4\mathrm{I}}$ extension of $\mathfrak{L}$ is a Cayley $D_4$.*

(4) *Every simple component of $\mathfrak{L}^*$ is a total matrix algebra over its center.*

*If $\Phi$ has characteristic* 0, *then each of the above is also equivalent to*

(5) *$\mathfrak{L}$ is a subalgebra of a Lie algebra of type $F_4$.*

(6) *$\mathfrak{L}$ is a subalgebra of a Lie algebra of type $E_{6\mathrm{I}}$.*

Theorem II. *If $\mathfrak{D}(\mathfrak{J}_1/\mathfrak{k}_1)$ and $\mathfrak{D}(\mathfrak{J}_2/\mathfrak{k}_2)$ are Jordan $D_4$'s, then $\mathfrak{D}(\mathfrak{J}_1/\mathfrak{k}_1) \cong \mathfrak{D}(\mathfrak{J}_2/\mathfrak{k}_2)$ if and only if there is a norm equivalence between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ which carries $\mathfrak{k}_1$ onto $\mathfrak{k}_2$. Moreover, if $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are split, then $\mathfrak{D}(\mathfrak{J}_1/\mathfrak{k}_1) \cong \mathfrak{D}(\mathfrak{J}_2/\mathfrak{k}_2)$ if and only if $\mathfrak{k}_1 \cong \mathfrak{k}_2$.*

Let $\mathfrak{J}$ be an exceptional central simple Jordan algebra over a field $E$, $\mathfrak{k}$ a cubic semisimple associative subalgebra of $\mathfrak{J}$, and suppose that $E/\Phi$ is finite-dimensional Galois. Then the groups $GL(\mathfrak{J}/\mathfrak{k})$ and $\Gamma L_\Phi(\mathfrak{J}/\mathfrak{k})$ are defined as the analogs of the corresponding groups discussed in Section 1. The elements of $GL(\mathfrak{J}/\mathfrak{k})$ induce automorphisms of $\mathfrak{L} = \mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ via conjugation and it will follow from later results that $GL(\mathfrak{J}/\mathfrak{k}) \to \mathrm{Aut}_E\, \mathfrak{L} \to 1$ is exact.

We let $K(\mathfrak{J}/\mathfrak{k})$ be the kernel so $1 \to K(\mathfrak{J}/\mathfrak{k}) \to GL(\mathfrak{J}/\mathfrak{k}) \to \mathrm{Aut}_E \mathfrak{L} \to 1$ is exact. If $\mathfrak{k}$ happens to be a diagonal algebra, then $K(\mathfrak{J}/\mathfrak{k}) = E^* \times E^* \times E^*$ as before and we will write $K_E$ instead of $K(\mathfrak{J}/\mathfrak{k})$.

The class of Jordan $D_4$'s contains $D_4$'s of every, type as we shall now see.

THEOREM 5. *Let $\mathfrak{L} = \mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ be a Jordan $D_4$ over $\Phi$. Then $\mathfrak{L}$ is of type*

(i) *$D_{4\mathrm{I}} \Leftrightarrow \mathfrak{k} = \Phi e_1 \oplus \Phi e_2 \oplus \Phi e_3$ is a diagonal algebra.*

(ii) *$D_{4\mathrm{II}} \Leftrightarrow \mathfrak{k} = \Phi e \oplus F$ where $e$ is an absolutely primitive idempotent and $F$ is the canonical $D_{4\mathrm{I}}$ field extension of $\mathfrak{L}$.*

(iii) *$D_{4\mathrm{III}} \Leftrightarrow \mathfrak{k}$ is a Galois field extension of $\Phi$ (the canonical $D_{4\mathrm{I}}$ field extension of $\mathfrak{L}$).*

(iv) *$D_{4\mathrm{VI}} \Leftrightarrow \mathfrak{k}$ is a non-Galois extension field of $\Phi$ (a canonical $D_{4\mathrm{II}}$ field extension of $\mathfrak{L}$).*

*Proof.* We will only establish one direction in each of the above, since the indicated cases are disjoint and exhaustive. Let $P/\Phi$ be a finite-dimensional Galois extension with $\mathfrak{J}_P$ split and $\mathfrak{k}_P = Pe_1 \oplus Pe_2 \oplus Pe_3$ a diagonal algebra. Without loss of generality we may assume that $\mathfrak{J}_P$ is the algebra $\tilde{\mathfrak{J}}$ of the preceeding sections and $e_1, e_2, e_3$ are the diagonal idempotents ([5], Section III, p. 81). Then $\mathfrak{L}_P = \mathfrak{D} = \mathfrak{D}(\tilde{\mathfrak{J}}/\sum Pe_i)$ is the split $D_4$.

The essential part of the proof is the determination of the precocycle of $G = g(P/\Phi)$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ corresponding to $\mathfrak{L}$. To this end let $s \to A(s)$ be the precocycle of $G$ in $\mathrm{Aut}_\Phi\, \tilde{\mathfrak{J}}$ corresponding to $\mathfrak{J}$. Since $\mathfrak{k} \subseteq \mathfrak{J}$, each $A(s)$ leaves $\mathfrak{k}_P = \sum Pe_i$ stable so $A(s) \in \Gamma L_\Phi(\tilde{\mathfrak{J}}/\sum Pe_i)$. Define $\eta(s) \in \mathrm{Aut}_\Phi\, \mathfrak{D}$ by $\eta(s) \leftrightarrow A(s)$. Then $A(s)\,A(t) = A(st)$ implies that $\eta(s)\,\eta(t) = \eta(st)$ so $s \to \eta(s)$ is a precocycle of $G$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ and corresponds to a $\Phi$-form of $\mathfrak{D}$. This $\Phi$-form clearly contains $\mathfrak{L}$, so it must be $\mathfrak{L}$.

$\mathfrak{k}$ is the fixed point set of $\sum Pe_i$ under $s \to A(s)|\sum Pe_i$ and is easily determined. If $\mathfrak{L}$ is of type $D_{4\mathrm{I}}$, then $e_iA(s) = e_i$ for all $s \in G$, $i = 1, 2, 3$; so $\mathfrak{k} = \sum \Phi e_i$, as stated. If $\mathfrak{L}$ is of type $D_{4\mathrm{II}}$, choose $i$ such that $ip(s) = i\,(A(s) = [p(s), T(s)])$. Then $e_iA(s) = e_i$ for all $s \in G$. If $i, j, k$ are unequal then $\alpha e_i + \beta e_j + \gamma e_k \in \mathfrak{k}$ if and only if $\alpha \in \Phi$, $\beta \in F$, and $\gamma = \beta^t$ where $t \mid F$ generates $g(F/\Phi)$. $\beta \to \beta e_j + \beta^t e_k$ is a monomorphism of $F$ into $\mathfrak{k}$ and $\mathfrak{k} = \Phi e \oplus F$, $e = e_i$.

If $\mathfrak{L}$ is of type $D_{4\mathrm{III}}$ then $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathfrak{k}$ if and only if $\alpha_1 \in F$, the canonical $D_{4\mathrm{I}}$-field extension of $\mathfrak{L}$, $\alpha_2 = \alpha_1^{s^2}$, and $\alpha_3 = \alpha_1{}^s$ where $p(s) = (123)$. $\alpha \to \alpha e_1 + \alpha^{s^2} e_2 + \alpha^s e_3$ is an isomorphism between $F$ and $\mathfrak{k}$. Since the last case is similar to the preceeding one, the proof is not given.

COROLLARY. *If $\mathfrak{L}$ is a Jordan $D_4$, then $\Gamma_P(\mathfrak{L}) = 1$.*

We next give the implication (2) $\Rightarrow$ (1) of Theorem I. There are several ways to do this. The one which we've chosen is more illuminating than the others and bears directly on (11).

Let $P/\Phi$ be a finite-dimensional Galois extension with Galois group $G$, $\mathfrak{J}$ an exceptional central simple Jordan algebra over $P$ and $\mathfrak{k}$ a cubic semisimple associative subalgebra of $\mathfrak{J}$. If $u \in \mathfrak{k}$ with $N(u) \neq 0$, we let $\mathfrak{J}^{(u)}$ be the $u$-isotope of $\mathfrak{J}$ ([*4*], [*5*]) and note that $\mathfrak{k}$ as a subspace of $\mathfrak{J}^{(u)}$ is again a cubic semisimple associative subalgebra. By Proposition 15 of [*4*], we see that $GL(\mathfrak{J}/\mathfrak{k}) = GL(\mathfrak{J}^{(u)}/\mathfrak{k})$ and $\Gamma L_\Phi(\mathfrak{J}/\mathfrak{k}) = \Gamma L_\Phi(\mathfrak{J}^{(u)}/\mathfrak{k})$. We will call $\mathfrak{J}^{(u)}$ ($u$ as above) a $\mathfrak{k}$-isotope of $\mathfrak{J}$. If $\mathfrak{M}$ is a $\Phi$-form of a $\mathfrak{k}$-isotope of $\mathfrak{J}$, and $\mathfrak{M}$ contains a $\Phi$-form $\mathfrak{N}$ of $\mathfrak{k}$, then we shall call $\mathfrak{M}$ a $\mathfrak{k}$-form of $\mathfrak{J}$ over $\Phi$. If $(\mathfrak{M}_1, \mathfrak{N}_1)$ and $(\mathfrak{M}_2, \mathfrak{N}_2)$ are $\mathfrak{k}$-forms of $\mathfrak{J}$ over $\Phi$ then we will call them norm equivalent if there is a norm equivalence between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which carries $\mathfrak{N}_1$ onto $\mathfrak{N}_2$. With these notions we have

THEOREM 6. *Suppose that $(\mathfrak{J}, \mathfrak{k})$ has a $\Phi$-form, i.e., $\mathfrak{J}$ has a $\Phi$-form which contains a $\Phi$-form of $\mathfrak{k}$. Then there is a* 1–1 *correspondence between $H^1(G, GL(\mathfrak{J}/\mathfrak{k}))$ and the set of norm equivalence classes of $\mathfrak{k}$-forms of $\mathfrak{J}$ over $\Phi$.*

*Proof.* Let $s \to C(s)$ be a precocycle of $G$ in $\Gamma L_\Phi(\mathfrak{J}/\mathfrak{k})$. The condition $C(st) = C(s)\, C(t)$ implies that $\{C(s)\}$ leaves fixed $(\mathfrak{M}, \mathfrak{N})$, a $\Phi$-form of the vector space pair $(\mathfrak{J}, \mathfrak{k})$. Since the restriction of the norm form on $\mathfrak{J}$ to $\mathfrak{k}$ is a nonzero polynomial function, there exists a $v \in \mathfrak{N}$ with $N(v) \neq 0$. We set $u = v^{-1}$ and suppose that $N(XC(s)) = \rho_s N(X)^s$. $vC(s) = v$ implies $\rho_s N(v)^s = N(v)$, so

$$N^{(u)}(XC(s)) = \rho_s N(v)^{-1}\, N(X)^s = (N(v)^{-1}\, N(X))^s = N^{(u)}(X)^s$$

([*4*], Proposition 15). This, together with $vC(s) = v$, implies that $C(s) \in \mathrm{Aut}_\Phi(\mathfrak{J}^{(u)}/\mathfrak{k})$ and thus $(\mathfrak{M}, \mathfrak{N})$ is a $\Phi$-form of $(\mathfrak{J}^{(u)}, \mathfrak{k})$, a $\mathfrak{k}$-isotope of $\mathfrak{J}$.

In the other direction, if $(\mathfrak{M}, \mathfrak{N})$ is a $\Phi$-form of a $\mathfrak{k}$-isotope of $\mathfrak{J}$, say $(\mathfrak{J}^{(u)}, \mathfrak{k})$, and $\mathfrak{N}$ is a $\Phi$-form of $\mathfrak{k}$, then this corresponds to a precocycle of $G$ in $\mathrm{Aut}_\Phi(\mathfrak{J}^{(u)}/\mathfrak{k}) \subseteq \Gamma L_\Phi(\mathfrak{J}/\mathfrak{k})$. Finally it is clear that the relation $B^{-1}C(s)\, B = C'(s)[B \in GL(\mathfrak{J}/\mathfrak{k})]$ holds between the precocycles $s \to C(s)$ and $s \to C'(s)$ if and only if $X \to XB$ is a norm equivalence between the associated $\mathfrak{k}$-forms.

COROLLARY. *If $\Gamma_P(\mathfrak{L}) = 1$, then $\mathfrak{L}$ is a Jordan $D_4$.*

*Proof.* If $\Gamma_P(\mathfrak{L}) = 1$ and $s \to \eta(s)$ is the precocycle of $G$ in $\mathrm{Aut}_\Phi \mathfrak{D}$ corresponding to $\mathfrak{L}$, then we may choose $C(s) \in \Gamma L_\Phi(\tilde{\mathfrak{J}}/\sum Pe_i)$ such that $\eta(s) \leftrightarrow C(s)$ and $C(st) = C(s)\, C(t)$. By the theorem, $\{C(s)\}$ leaves fixed $(\mathfrak{M}, \mathfrak{N})$, a $\sum Pe_i$-form of $\tilde{\mathfrak{J}}$ over $\Phi$. It is immediate that $\mathfrak{L} = \mathfrak{D}(\mathfrak{M}/\mathfrak{N})$.

We are now in a position to prove

THEOREM 7. *Let* $\mathfrak{L}_1 = \mathfrak{D}(\mathfrak{J}_1/\mathfrak{k}_1)$ *and* $\mathfrak{L}_2 = \mathfrak{D}(\mathfrak{J}_2/\mathfrak{k}_2)$ *be Jordan* $D_4$'s *over* $\Phi$. *Then any isomorphism between them has the form* $D_1 \to B^{-1}D_1B$, *where* $B$ *is a norm equivalence from* $\mathfrak{J}_1$ *to* $\mathfrak{J}_2$ *which carries* $\mathfrak{k}_1$ *onto* $\mathfrak{k}_2$. *Conversely, any correspondence of this form is an isomorphism between* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$.

*Proof.* Let $P/\Phi$ be a finite-dimensional Galois splitting extension for $\mathfrak{L}_i$, $i = 1, 2$. Without loss of generality we may assume $\mathfrak{J}_{iP} = \tilde{\mathfrak{J}}$, the algebra of Section 1; and $\mathfrak{k}_{iP} = \sum Pe_i$ the diagonal algebra of Section 1. Let $s \to \eta_i(s) \leftrightarrow A_i(s)$ be the precocycle of $G = g(P/\Phi)$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ corresponding to $\mathfrak{L}_i$, $i = 1, 2$, which was introduced in the proof of Theorem 5. Any isomorphism $\beta : \mathfrak{L}_1 \to \mathfrak{L}_2$ extends to an automorphism of $\mathfrak{D}$ which we again denote by $\beta$. If $\beta \leftrightarrow D \in GL(\tilde{\mathfrak{J}}/\sum Pe_i)$, then the relation $\beta^{-1}\eta_1(s)\,\beta = \eta_2(s)$ implies that $D^{-1}A_1(s)\,D = A_2(s)\,\mu_s$, $\mu_s \in K$. Using the action on $K$ afforded by $\mathfrak{L}_2$ we see that $s \to \mu_s$ is a 1-cocycle of $G$ in $K$. As remarked earlier $H^1(G, K) = 1$, so $\mu_s$ has the form $\mu_s = \lambda^{-1}\lambda^s$ for some $\lambda \in K$. Set $B = D\lambda$. Then $B^{-1}A_1(s)\,B = \lambda^{-1}D^{-1}A_1(s)\,D\lambda = \lambda^{-1}A_2(s)\,\mu_s\lambda = A_2(s)$, which shows that $B \mid \mathfrak{J}_1$ is a norm equivalence between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ which carries $\mathfrak{k}_1$ onto $\mathfrak{k}_2$. Since $\beta \leftrightarrow B$, $\beta : D_1 \to B^{-1}D_1B$, as stated. Since the proof of the second assertion is similar to that of the second part of Theorem 2, it is omitted.

COROLLARY. *Let* $\mathfrak{L}_1 = \mathfrak{D}(\mathfrak{J}_1/\mathfrak{k}_1)$ *and* $\mathfrak{L}_2 = \mathfrak{D}(\mathfrak{J}_2/\mathfrak{k}_2)$ *be Jordan* $D_4$'s *with* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *split. Then* $\mathfrak{L}_1 \cong \mathfrak{L}_2$ *if and only if* $\mathfrak{k}_1 \cong \mathfrak{k}_2$.

*Proof.* The proof of Theorem 5 shows that $\mathfrak{k}_i \cong$ center $\mathfrak{L}_i^*$. Thus if $\mathfrak{L}_1 \cong \mathfrak{L}_2$ then also $\mathfrak{L}_1^* \cong \mathfrak{L}_2^*$ and hence $\mathfrak{k}_1 \cong \mathfrak{k}_2$. For the converse we must distinguish between the various $D_4$ types.

If $\mathfrak{L}_i$ is of type $D_{4\mathrm{I}}$, i.e., $\mathfrak{k}_i$ is a diagonal algebra, then the proof of Theorem 1 shows that $\mathfrak{L}_i$ is the split $D_4$ so $\mathfrak{L}_1 \cong \mathfrak{L}_2$. If $\mathfrak{L}_i$ is exceptional [i.e., $\mathfrak{k}_i$ is a field], let $\mathfrak{k}_i = \Phi(a_i)$. Since we are assuming that $\mathfrak{k}_1 \cong \mathfrak{k}_2$ we may assume that our isomorphism carries $a_1$ onto $a_2$. By a theorem of Jacobson ([5], Section III, Theorem 6) we may embed $a_i$ [hence $\mathfrak{k}_i$] in a subalgebra $\mathfrak{B}_i$ of $\mathfrak{J}_i$, where $\mathfrak{B}_i \cong \Phi_3{}^+$. The isomorphism of $\mathfrak{k}_1$ onto $\mathfrak{k}_2$ determined by $a_1 \to a_2$ may be extended to an isomorphism of $\mathfrak{B}_1$ onto $\mathfrak{B}_2$. This in turn ([1], p. 412) may be extended to an isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Such an isomorphism is necessarily a norm equivalence of $\mathfrak{J}_1$ onto $\mathfrak{J}_2$ carrying $\mathfrak{k}_1$ onto $\mathfrak{k}_2$. By the theorem, $\mathfrak{L}_1 \cong \mathfrak{L}_2$. The demonstration for the case when $\mathfrak{L}_i$ is of type $D_{4\mathrm{II}}$ is similar to the above.

We conclude this section by showing that $\mathfrak{L}$ is a Jordan $D_4$ if and only if the canonical $D_{4\mathrm{I}}$ extension of $\mathfrak{L}$ is a Cayley $D_4$. One direction is clear since the proof of Theorem 1 shows that the class of Jordan $D_{4\mathrm{I}}$'s coincides with the class of Cayley $D_4$'s.

For the moment suppose that $E/\Phi$ is a finite-dimensional Galois extension with $\mathfrak{L}_E = \mathfrak{D}(\mathfrak{J}/\mathfrak{k})$, a Jordan $D_4$. We will develop a criterion over $E$ for determining whether or not $\mathfrak{L}$ is a Jordan $D_4$.

Let $P/\Phi \supseteq E/\Phi$ be a finite-dimensional Galois splitting extension for $\mathfrak{L}$. We may take $\mathfrak{J}_P = \tilde{\mathfrak{J}}$ and $\mathfrak{k}_P = \sum Pe_i$ as before. Let $H$ be the subgroup of $G = g(P/\Phi)$ corresponding to $E$. If $s \to \eta(s)$ is the precocycle of $G$ in $\mathrm{Aut}_\Phi\, \mathfrak{D}$ determined by $\mathfrak{L}$, then our condition on $\mathfrak{L}_E$ enables to assume that $\eta(h) \leftrightarrow A(h)$ for $h \in H$, where $h \to A(h)$ is the precocycle of $H$ in $\mathrm{Aut}_E(\tilde{\mathfrak{J}}/\sum Pe_i)$ corresponding to $(\mathfrak{J}, \mathfrak{k})$.

Let $r \in G$, $r \notin H$ and choose $C(r) \in \Gamma L_\Phi(\tilde{\mathfrak{J}}/\sum Pe_i)$, such that $\eta(r) \leftrightarrow C(r)$. For any $r \in G$, $h \in H$, $r^{-1}hr = h^r \in H$ so $\eta(r)^{-1}\,\eta(h)\,\eta(r) = \eta(h^r)$. This implies that $C(r)^{-1}A(h)\,C(r) = A(h^r)\,\mu_{h^r}$ where $\mu_{h^r} \in K$. $A(hh') = A(h)\,A(h')$ for all $h$, $h'$ in $H$ implies that $h \to \mu_h$ is a 1-cocycle of $H$ in $K$ and hence has the form $\mu_h = \lambda^{-1}\lambda^h$ for some $\lambda \in K$. Set $D(r) = C(r)\,\lambda$. Then $\eta(r) \leftrightarrow D(r)$ and $D(r)^{-1}\,A(h)\,D(r) = A(h^r)$. Let $r \to \bar{r}$ be the canonical homomorphism of $G \to G/H \cong g(E/\Phi)$. Our last equation implies that $D(\bar{r}) = D(r)|\,\mathfrak{J}$ is an $\bar{r}$-linear norm equivalence of $\mathfrak{J}$ leaving $\mathfrak{k}$ stable.

Let $\bar{r} \to \eta(\bar{r})$ be the precocycle of $G/H$ in $\mathrm{Aut}_\Phi\, \mathfrak{L}_E$ leaving $\mathfrak{L}$ fixed. The preceeding remarks show that $\eta(\bar{r}) : D \to D(\bar{r})^{-1}\,DD(\bar{r})$ and hence we have a relation of the form $D(\bar{r})\,D(\bar{r}') = D(\bar{r}\bar{r}')\,\delta_{\bar{r},\bar{r}'}$ where $\delta_{\bar{r}\bar{r}'} \in K(\mathfrak{J}/\mathfrak{k})$. If we can choose $C(\bar{r}) \in \Gamma L_\Phi(\mathfrak{J}/\mathfrak{k})$ such that $C(\bar{r})\,C(\bar{r}') = C(\bar{r}\bar{r}')$ and $\eta(\bar{r}) \leftrightarrow C(\bar{r})$, then the proof of the corollary of Theorem 6 shows that $\mathfrak{L}$ is a Jordan $D_4$. Another way of seeing this is to note that

$$(\bar{r}, \bar{r}') \to \delta_{\bar{r}\bar{r}'} \in K(\mathfrak{J}/\mathfrak{k}) \subseteq K$$

is a two-cocycle of $G/H$ in $K(\mathfrak{J}/\mathfrak{k})$. Then, if $[\bar{\delta}]$ denotes its two-cohomology class, we have $\Gamma_P(\mathfrak{L}) = \text{inflation}_{\,G/H,G}[\bar{\delta}]$. The condition which we have supposed satisfied is just $[\bar{\delta}] = 1$, so trivially $\Gamma_P(\mathfrak{L}) = 1$. This is the criterion which we shall use.

Suppose now that $\mathfrak{L}_F$ is a Jordan $D_4$, $F$ the canonical $D_{4\mathrm{I}}$-field extension of $\mathfrak{L}$. We must compute some two-cohomology classes. When $\mathfrak{L}$ is of type $D_{4\mathrm{II}}$ or $D_{4\mathrm{III}}$, this is accomplished in the next lemma. If $\mathfrak{L}$ is of type $D_{4\mathrm{VI}}$ the computation is more complicated and is given separately.

In what follows, $E/\Phi$ is a suitable Galois extension, $\mathfrak{J}$ is a reduced exceptional central simple Jordan algebra and $\{e_1, e_2, e_3\}$ a set of complimentary orthogonal idempotents.

Lemma 2. (i) *If $C \in \Gamma L_\Phi(\mathfrak{J}/\sum Ee_i)$ is t-linear, $t^2 = 1$, $t \neq 1$, $C = [(12), U]$, and $C^2 \in K_E$, then there is a $\mu \in K_E$ with $(C\mu)^2 = 1$.*

(ii) *If $C \in \Gamma L_\Phi(\mathfrak{J}/\sum Ee_i)$ is s-linear, $s^3 = 1$, $s \neq 1$, $C = [(123), T]$, and $C^3 \in K_E$, then there is a $\mu \in K_E$ with $(C\mu)^3 = 1$.*

*Proof.* Without loss of generality we may suppose that $C$ is norm-preserving.

(i) Write $C^2 = \delta = (\delta_i) \in K_E$. Then $\delta^t = C^{-1}\delta C = \delta$ so $(\delta_2{}^t, \delta_1{}^t, \delta_3{}^t) = (\delta_1, \delta_2, \delta_3)$ and we obtain, $\delta_1{}^t = \delta_2$ and $\delta_3{}^t = \delta_3 = (\delta_1\delta_1{}^t)^{-1}$. Take $\mu = (\delta_1^{-1}, 1, \delta_1)$. Then

$$(C\mu)^2 = \delta\mu^t\mu = (\delta_1, \delta_1{}^t, (\delta_1\delta_1{}^t)^{-1})(1, (\delta_1{}^t)^{-1}, \delta_1{}^t)(\delta_1^{-1}, 1, \delta_1) = 1.$$

(ii) Let $C^3 = \delta = (\delta_i) \in K_E$. $\delta^s = C^{-1}\delta C = \delta$, so $(\delta_2{}^s, \delta_3{}^s, \delta_1{}^s) = (\delta_1, \delta_2, \delta_3)$. This yields $\delta_2 = \delta_1^{s^2}$ and $\delta_3 = \delta_1{}^s$. Since $\delta_1\delta_2\delta_3 = 1$ we see that $\delta_1\delta_1{}^s\delta_1^{s^2} = 1$, so $\delta_1 = \lambda_1(\lambda_1^{-1})^s$ for some $\lambda_1 \in E^*$. If we set $\mu = (\lambda_1^{-1}, \lambda_1, 1)$, then

$$(C\lambda)^3 = \delta\mu\mu^s\mu^{s^2} = (\delta_1, \delta_1^{s^2}, \delta_1{}^s)(\lambda_1^{-1}, \lambda_1, 1)(\lambda_1{}^s, 1, (\lambda_1{}^s)^{-1})(1, (\lambda_1^{s^2})^{-1}, \lambda_1^{s^2}) = 1.$$

LEMMA 3. *Let $\mathfrak{L}$ be a Lie algebra of type $D_{4\text{VI}}$ over $\Phi$, $F$ its canonical $D_{4\text{I}}$-field extension. Then $\mathfrak{L}$ is a Jordan $D_4$ if $\mathfrak{L}_F$ is.*

*Proof.* Write $\mathfrak{L}_F = \mathfrak{D}(\mathfrak{J}/\sum Fe_i)$ and let $r \to \eta(r) \leftrightarrow [p(r), T(r)] = C(r) \in \Gamma L_\Phi(\mathfrak{J}/\sum Fe_i)$ be the precocycle of $G = g(F/\Phi)$ in $\mathrm{Aut}_\Phi \mathfrak{L}_F$ corresponding to $\mathfrak{L}$. $r \to p(r)$ is an antiisomorphism between $G$ and $S_3$ and we choose generators $t, s$ for $G$ with $p(t) = (12)$ and $p(s) = (123)$.

By the preceeding lemma we may assume that $C(t)^2 = C(s)^3 = 1$. $\eta(s)\,\eta(t) = \eta(t)\,\eta(s)^2$ implies that $C(s)\,C(t) = C(t)\,C(s)^2\,\omega$ for some $\omega \in K_F$. Solving for $C(t)$ we obtain $C(t) = C(s)^2\,C(t)\,C(s)^2\,\omega = C(s)\,C(t)\,C(s)\,\omega^{s^2}\,\omega = C(t)\,\omega\omega^s\omega^{s^2}$ $[\delta^r = C^{-1}(r)\delta C(r)]$, so $1 = \omega\omega^s\omega^{s^2}$. Similarly solving for $C(s)$ we obtain $1 = \omega\omega^t\omega^{s^2}$. Thus $\omega^s = \omega^t$. Now set $D(s) = C(s)\,\omega$ and $D(t) = C(t)$. Then $D(s)^3 = \omega\omega^s\omega^{s^2} = 1 = D(t)^2$ and $D(s)\,D(t) = C(s)\,C(t)\,\omega^t = C(t)\,C(s)^2\,\omega\omega^s = D(t)\,D(s)^2$. These relations show that $t \to D(t)$ and $s \to D(s)$ determine a precocycle of $G$ in $\Gamma L_\Phi(\mathfrak{J}/\sum Fe_i)$. Thus $\mathfrak{L}$ is a Jordan $D_4$.

COROLLARY *Let $\mathfrak{L}$ be a Lie algebra of type $D_{4\text{VI}}$ over $\Phi$. Then $\mathfrak{L}$ is a Jordan $D_4$ if and only if its canonical $D_{4\text{III}}$ extension is a Jordan $D_4$.*

*Proof.* Both algebras have the same canonical $D_{4\text{I}}$ extension.

## 4. SPECIAL FIELDS

Let $\Phi$ be a finite field. The only central simple (finite-dimensional) division algebra over $\Phi$ is $\Phi$-itself, i.e., every central simple associative algebra over $\Phi$ is split. Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over $\Phi$, $P/\Phi$ a finite-dimensional Galois splitting extension and $F/\Phi$ the canonical $D_{4\text{I}}$-field extension of $\mathfrak{L}$.

If $\Gamma_P(\mathfrak{L}_F)$ is the cohomology class of the two-cocycle $(h, t) \to \delta_{h,t} = (\delta_{i(h,t)})$, then the factor sets $(h, t) \to \delta_{i(h,t)}$ correspond to the central simple factors of $\mathfrak{L}_F{}^*$. Since each $\delta_i$ is split, it follows that $\delta$ is split. Hence $\mathfrak{L}_F$ is a Jordan $D_4$. By Theorem I, $\mathfrak{L}$ is a Jordan $D_4$. The only exceptional central simple Jordan algebra over $\Phi$ is split, so $\mathfrak{L}$ has the form $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ where $\mathfrak{J}$ is split. The corollary to Theorem 7 gives the isomorphism condition for such algebras. Hence there are exactly three $D_4$'s over $\Phi$, the split $D_4$ and a $D_{4\text{II}}$ and $D_{4\text{III}}$ corresponding to the unique quadratic and cubic overfields of $\Phi$.

Now let $\Phi$ be a finite algebraic extension of a $p$-adic field. The only finite-dimensional central simple involutorial associative algebras over $\Phi$ are $\Phi_n$ and $Q_n$ where $Q$ is the unique quaternion division algebra over $\Phi$ [9]. If $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are the central simple components of the enveloping algebra of a Lie algebra of type $D_{4\text{I}}$ (over $\Phi$) then the relation $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{A}_3 \sim 1$ implies that at least one of the $\mathfrak{A}_i$ is split. Thus if $\mathfrak{L}_F$ is the canonical $D_{4\text{I}}$ extension of an exceptional Lie algebra of type $D_4$ over $\Phi$, then by the remark preceeding the discussion of $\Gamma_P(\mathfrak{L})$ in Section 2, we see that $\mathfrak{L}_F{}^*$ is a sum of total matrix algebras. As in the preceeding case this implies that $\mathfrak{L}_F$ is a Jordan $D_4$, hence $\mathfrak{L}$ is also. One sees quite easily that the only Jordan $D_{4\text{I}}$ over $\Phi$ is the split $D_4$. By using Lemma 2 one also concludes that the set of Jordan $D_{4\text{II}}$'s over $\Phi$ coincides with the set of algebras of the form $\mathfrak{S}(\Phi_8, J)$ where $J$ is an involution corresponding to a nondegenerate quadratic form of Witt index 3.

Again the only exceptional central simple Jordan algebra over $\Phi$ is split. Using the corollary to Theorem 7 we see that there is a 1–1 correspondence between the set of Jordan $D_4$'s over $\Phi$ and the set of overfields of $\Phi$ of, at most, degree 3.

## 5. Further Results

The equivalence of (2) and (4) of Theorem I involves the identification of $H^2(G, K)$ with classes of $\Phi$-forms of $P_n \oplus P_n \oplus P_n$. At the present time, the ramifications of this correspondence and its general importance have not been determined, so we have excluded it from our discussion. However we shall note the following: if $\mathfrak{X}$ is a $\Phi$-form of $P_n \oplus P_n \oplus P_n$ then its correspondent in $H^2(G, K)$ is 1 if and only if every simple component of $\mathfrak{X}$ is a total matrix algebra over its center. This generalizes the result indicated at the beginning of Section 4.

The equivalence of (1) and (5) of Theorem I can be seen from the conjugacy of $D_4$ subalgebras of a Lie algebra of type $F_4$ over an algebraically closed field of characteristic 0. The equivalence of (1) and (6) is similar, and follows from the conjugacy of $D_4$ subalgebras of a Lie algebra of type $E_6$ over an algebraically closed field of characteristic 0. This last result is due to J. Ferrar [*3*].

Finally we have a partial result for $D_4$'s over finite algebraic number fields. We have

THEOREM 8. *Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over an algebraic number field $\Phi$. Then there exists at most a finite number of primes* p *on $\Phi$ such that $\mathfrak{L}_{\Phi_\mathfrak{p}}$ is not a Jordan $D_4$. Moreover, $\mathfrak{L}$ is a Jordan $D_4$ if and only if $\mathfrak{L}_{\Phi_\mathfrak{p}}$ is a Jordan $D_4$ for every prime* p *on $\Phi$.*

Using this result one can show that $\mathfrak{L}$ is split by a Galois extension of, at most, degree $2[F:\Phi]$ where $F$ is the canonical $D_{4\mathrm{I}}$-field extension of $\mathfrak{L}$.

## ACKNOWLEDGMENT

The author would like to take this opportunity to thank Professor Nathan Jacobson who suggested this work and whose advice and encouragment has been invaluable.

## REFERENCES

*1*. ALBERT, A. A. AND JACOBSON, N. Reduced exceptional simple Jordan algebras. *Ann. Math.* **67** (1957), 400-417.
*2*. CHEVALLEY, C. AND SCHAFER, R. D. The exceptional central simple Lie algebras $F_4$ and $E_6$. *Proc. Natl. Acad. Sci. U.S.* **36** (1950), 137-141.
*3*. FERRAR, J. C. On Lie algebras of type $E_6$. Yale Univ. dissertation (in preparation).
*4*. JACOBSON, N. Exceptional Lie Algebras. Yale Univ. mimeographed Report.
*5*. JACOBSON, N. Some groups of transformations defined by Jordan algebras, I, II, and III. *J. Reine Angew. Math.* **201** (1959), 178-195; **204** (1960), 74-98; and **204** (1961), 61-85.
*6*. JACOBSON, N. "Lie Algebras." Interscience, New York, 1962.
*7*. JACOBSON, N. Triality and Lie algebras of type $D_4$. *Rend. Circ. Palermo, Ser. II* **13** (1964), 1-25.
*8*. JACOBSON, N. Forms of algebras, "Proceedings of the 3rd Annual Science Conference of the Belfer Graduate School of Science." Yeshiva University, New York.
*9*. O'MEARA, O. T. "An Introduction to Quadratic Forms." Springer, Berlin, 1963.
*10*. SELIGMAN, G. B. On automorphisms of Lie algebras of classical type, I, II, and III. *Trans. Am. Math. Soc.* **92** (1959), 430-448; **94** (1960), 286-316; and **97** (1960), 452-482.
*11*. SERRE, J. P. "Corps Locaux." Hermann, Paris, 1962.
*12*. SPRINGER, T. A. On the equivalence of quadratic forms. *Proc. Koninkl. Ned. Akad. Wetenschap., Ser. A* **69** (1959), 241-253.
*13*. SPRINGER, T. A. AND VAN DER BLIJ, F. Octaves and triality. *Nieuw Arch. Wiskunde, Ser. 3* **8** (1960), 158-169.

*Printed in Belgium*